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Quantum Hall effect wavefunctions as cyclic representations of $U_q(sl(2))$

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Abstract. Quantum Hall effect wavefunctions corresponding to the filling factors $1/2p+1, 2/2p+1, \ldots, 2p/2p+1, 1$, are shown to form a basis of irreducible cyclic representations of the quantum algebra $U_q(sl(2))$ at $q^{2p+1}=1$. Thus, the wavefunctions $\Psi_{P/Q}$ possessing filling factors $P/Q<1$ where Q is odd and P, Q are relatively prime integers are classified in terms of $U_q(sl(2))$.

1. Introduction

The microscopic theory of the fractional quantum Hall effect (QHE) is not well established. Its theoretical understanding is mostly due to trial wavefunctions [1]. For filling factors $1/m$ where m is an odd integer, trial wavefunctions were given by Laughlin [2]. Trial wavefunctions for the other filling factors $\nu = P/Q < 1$, where P, Q are relatively prime integers and Q is odd, were constructed in terms of some hierarchy schemes [3, 4] where they were obtained from a parent state which is a fulfilled Landau level or a Laughlin wavefunction. However, general properties of the QHE should be independent of the explicit form of trial wavefunctions, but depend on their universal features as their orthogonality.

We utilize orthogonality of QHE states for different filling factors, independent of their explicit form, to show that they can be classified as irreducible cyclic representations of $U_q(sl(2))$ at roots of unity. In our scheme, states corresponding to filling factors possessing a common denominator are in the same representation.

Although $U_q(sl(2))$ structures were found in the Hofstadter problem [5], in the Landau problem [6], for Laughlin wavefunctions [7] and in the QHE [8], the approach presented here does not have any relation to them. (i) In all of the previous works dealing with flat surfaces, generators of the deformed algebra were constructed in terms of magnetic translations. The construction presented here cannot be written in terms of magnetic transformations. (ii) Here, wavefunctions possessing different filling factors which have a common denominator are treated on the same footing. However, in the other works only one state is considered and the theories were built on them without mixing different states with different parameters which correspond to filling factors in the QHE case.

First, we show explicitly that the wavefunctions corresponding to the filling factors $\nu = \frac{1}{3}, \frac{2}{3}, 1$, can be considered as the basis of cyclic irreducible representations of the quantum algebra $U_q(sl_2)$ at $q^3 = 1$. Then, the general case is studied. Conclusions are presented in the final section.

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2. Cyclic representation of $U_q(sl(2))$

The deformed algebra $U_q(sl(2))$

$$
\begin{aligned}
&[E_+, E_-] = \frac{K - K^{-1}}{q - q^{-1}} \\
&K E_\pm K^{-1} = q^{\pm 2} E_\pm
\end{aligned}
\tag{1}
$$

at roots of unity, i.e. $q^{2p+1} = 1$, p is a positive integer, has a finite-dimensional irreducible representation which has no classical finite-dimensional analogue. This is the cyclic representation whose dimension is $2p+1$ [9]. Cyclic means that there are no highest or lowest weight states in the spectrum, i.e. $E_+|\ldots\rangle \neq 0$ and $E_-|\ldots\rangle \neq 0$ for any state.

When $q^{2p+1} = 1$ irreducible cyclic representation of $U_q(sl(2))$ can be written in some basis $\{v_0, v_1, \ldots, v_{2p}\}$ as

$$
\begin{aligned}
&K v_m = \lambda q^{-2m} v_m \\
&E_+ v_m = g_m v_{m+1} \\
&E_- v_m = f_m v_{m-1}
\end{aligned}
\tag{2}
$$

where $m = 0, \ldots, 2p$, and we defined $v_0 \equiv v_{2p+1}$, $v_{-1} \equiv v_{2p}$. λ, g_m and f_m are some complex constants which are nonzero and in the case of requesting that the representation in unitary, we should restrict their values such that

$$
K^\dagger = K^{-1} \qquad E_-^\dagger = E_+. \tag{3}
$$

Although, for the purposes of this work there is no need to discuss in detail neither how unitary representations arise in the general framework nor values of Casimir operators, let us denote that there are three independent Casimir operators of $U_q(sl(2))$ at $q^{2p+1} = 1$: K^{2p+1}, E_+^{2p+1} and E_-^{2p+1}.

3. Classification of $\nu = 1, \frac{1}{3}, \frac{2}{3}$ states

When N particles (electrons) move on a plane in a perpendicular magnetic field we may consider the wavefunctions [2, 10]

$$
\psi_1(z_1, \ldots, z_N) = \mathcal{N}_1 \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^N |z_k|^2} \prod_{i<j}^N (z_i - z_j) \tag{4}
$$

$$
\psi_{1/3}(z_1, \ldots, z_N) = \mathcal{N}_2 \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^N |z_k|^2} \prod_{i<j}^N (z_i - z_j)^3 \tag{5}
$$

$$
\psi_{2/3}(z_1, \ldots, z_N) = \mathcal{N}_3 \int \mathrm{d}^2 z_{N+1} \ldots \mathrm{d}^2 z_{N+M} \mathrm{e}^{-\frac{1}{2}\sum_{K=1}^{N+M} |z_K|^2} \prod_{l<n}^M (\bar{z}_{N+l} - \bar{z}_{N+n})^3 \prod_{I<J}^{N+M} (z_I - z_J) \tag{6}
$$

which possess the following values of the angular momentum L,

$$
\begin{aligned}
&L[\psi_1(z_1, \ldots, z_N)] = \frac{N(N-1)}{2} \\
&L[\psi_{1/3}(z_1, \ldots, z_N)] = 3\frac{N(N-1)}{2} \\
&L[\psi_{2/3}(z_1, \ldots, z_N)] = \frac{(N+M)(N+M-1)}{2} - 3\frac{M(M-1)}{2}.
\end{aligned}
$$

It is supposed that N is large and we take $M = N/2$.

Filling factors of the N particle states are given in the thermodynamical limit as

$$\nu \equiv \lim_{N\to\infty} \frac{N(N-1)}{2L}. \tag{7}$$

Hence, filling factors of the wavefunctions (4)–(6) are

$$\nu(\psi_1) = 1 \qquad \nu(\psi_{1/3}) = \tfrac{1}{3} \qquad \nu(\psi_{2/3}) = \tfrac{2}{3}. \tag{8}$$

Indeed, (4) is the wavefunction when the lowest Landau level is fully filled and (5) and (6) are the trial wavefunctions which describe the QHE at the filling factors $\frac{1}{3}, \frac{2}{3}$.

By using

$$\int \mathrm{d}^2 z\, \mathrm{e}^{-|z|^2} \bar{z}^m z^n = \delta_{m,n} \tag{9}$$

one can observe that the wavefunctions which possess different angular momentum values are orthogonal. Moreover, by choosing the normalization constants $\mathcal{N}_a$ appropriately the wavefunctions (4)–(6) can be taken to satisfy ($N > 2$)

$$(\psi_\sigma, \psi_\rho) \equiv \int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_N\, \bar{\psi}_\sigma(z_1, \ldots, z_N) \psi_\rho(z_1, \ldots, z_N) = \delta_{\sigma,\rho} \tag{10}$$

where $\sigma, \rho = 1, \frac{1}{3}, \frac{2}{3}$.

If $\hat{\nu}$ denotes the first quantized operator corresponding to the filling factor ν, one can construct the physical operator

$$\hat{k} \equiv \mathrm{e}^{2\pi \mathrm{i} \hat{\nu}} \tag{11}$$

which will be shown to play the main role in classifying QHE wavefunctions in terms of $U_q(sl(2))$ at roots of unity.

In a second quantized theory, operators corresponding to physical operators of the first quantization will be given in terms of states spanning the related field theory. Let us deal with the states, corresponding to (4)–(6),

$$|\sigma\rangle \equiv \int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_N\, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^N |z_k|^2} \psi_\sigma(z_1, \ldots, z_N) |z_1, \ldots, z_N\rangle \tag{12}$$

where

$$|z_1, \ldots, z_N >= \frac{1}{\sqrt{N!}} \varphi^\dagger(z_1) \ldots \varphi^\dagger(z_N)|0\rangle. \tag{13}$$

The fermionic operators $\varphi(z), \varphi^\dagger(z)$ satisfy the anticommutation relation

$$\{\varphi^\dagger(z), \varphi(z')\} = \mathrm{e}^{z'\bar{z}}.$$

The states (12) are orthonormal:

$$\langle\sigma|\rho\rangle = \delta_{\sigma,\rho}. \tag{14}$$

The second quantized operator

$$k = \mathrm{e}^{2\pi \mathrm{i}} |1\rangle\langle 1| + \mathrm{e}^{2\pi \mathrm{i}/3} |\tfrac{1}{3}\rangle\langle\tfrac{1}{3}| + \mathrm{e}^{4\pi \mathrm{i}/3} |\tfrac{2}{3}\rangle\langle\tfrac{2}{3}| \tag{15}$$

corresponds to the first quantized physical operator (11). In terms of the vector $(|1\rangle, |\frac{1}{3}\rangle, |\frac{2}{3}\rangle)$ and the scalar product defined in (14), one can obtain the representation

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{q} & 0 \\ 0 & 0 & \tilde{q}^2 \end{pmatrix} \tag{16}$$

where $\tilde{q} = \exp(2\pi \mathrm{i}/3)$, i.e.

$$\tilde{q}^3 = 1.$$

Moreover, we can construct the operators

$$e_+ = a_1|\tfrac{1}{3}\rangle\langle 1| + a_2|\tfrac{2}{3}\rangle\langle\tfrac{1}{3}| + a_3|1\rangle\langle\tfrac{2}{3}| \tag{17}$$

$$e_- = b_1|1\rangle\langle\tfrac{1}{3}| + b_2|\tfrac{1}{3}\rangle\langle\tfrac{2}{3}| + b_3|\tfrac{2}{3}\rangle\langle 1| \tag{18}$$

whose representations are

$$e_+ = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{pmatrix} \qquad e_- = \begin{pmatrix} 0 & 0 & b_3 \\ b_1 & 0 & 0 \\ 0 & b_2 & 0 \end{pmatrix}. \tag{19}$$

One can show that (16) and (19) realize the $U_q(sl(2))$ algebra

$$[e_+, e_-] = \frac{k - k^{-1}}{\tilde{q} - \tilde{q}^{-1}} \qquad k e_\pm k^{-1} = \tilde{q}^{\pm 2} e_\pm \tag{20}$$

if the coefficients satisfy

$$\begin{aligned} a_1 b_1 - a_3 b_3 &= 0 \\ a_2 b_2 - a_1 b_1 &= 1 \\ a_3 b_3 - a_2 b_2 &= -1. \end{aligned} \tag{21}$$

If one demands that the representation (16), (19) is unitary:

$$k^{-1} = k^\dagger \qquad {e_+}^\dagger = e_- \tag{22}$$

the coefficients a_l, b_l should be taken as

$$b_1 = \bar{a}_1 \qquad b_2 = \bar{a}_2 \qquad b_3 = \bar{a}_3. \tag{23}$$

Then conditions (21) lead to

$$|a_1|^2 = |a_3|^2 = |a_2|^2 - 1 \qquad a_l \neq 0. \tag{24}$$

Observe that the Casimir operators

$$k^3 = \mathbf{1} \qquad e_+^3 = a_1 a_2 a_3 \mathbf{1} \qquad e_-^3 = b_1 b_2 b_3 \mathbf{1}$$

are proportional to identity.

An explicit realization is presented in terms of the trial wavefunctions (4)–(6). However, the construction depends only on the orthogonality of the states of the QHE for different values of the filling factors and the existence of the physical operator (11). This will be clarified in the next section.

4. The general case

QHE trial wavefunctions in the standard hierarchy scheme are given by [3, 11]

$$\begin{aligned} \psi_\nu(z_1, \ldots, z_{N_0}) = \int \prod_{\alpha=1}^{r} \prod_{i_\alpha=1}^{N_\alpha} [\mathrm{d}^2 z_{i_\alpha}^{(\alpha)}] \mathrm{e}^{-\frac{1}{2}\sum_1^{N_0} |z_k|^2} \prod_{\beta=0}^{r} \prod_{i_\beta < j_\beta}^{N_\beta} (z_{i_\beta}^{(\beta)} - z_{j_\beta}^{(\beta)})^{a_\beta} \\ \times \prod_{i_{\beta+1}, j_\beta = 1}^{N_{\beta+1}, N_\beta} (z_{i_{\beta+1}}^{(\beta+1)} - z_{j_\beta}^{(\beta)})^{b_{\beta,\beta+1}} \end{aligned} \tag{25}$$

where $z_{i_0}^{(0)} \equiv z_i$. The measure $\prod[\mathrm{d}^2 z_{i_\alpha}^{(\alpha)}]$ depends on a_β and $|z_{i_\beta}^{(\beta)} - z_{j_\beta}^{(\beta)}|$, however, the detailed form of it does not affect the filling factor $\nu = P/Q$. a_0 is an odd positive integer,

a_α for $\alpha \neq 0$ are even integers which can be positive or negative and $b_{\beta+1,\beta} = \pm 1$, except $b_{r,r+1} = 0$. By placing the N_0 electrons on a spherical surface in a monopole magnetic field, one can find that filling factor of (25) is given by

$$\nu = \cfrac{1}{a_0 - \cfrac{1}{a_1 - \cfrac{1}{\cdots - \cfrac{1}{a_r}}}}. \tag{26}$$

Factors with negative powers may be replaced by complex-conjugate factors with positive powers multiplied by some exponential factors. Hence, (25) can be equivalently given as [12]

$$\psi_\nu(z_1, \ldots, z_{N_0}) = \int \prod_{\alpha=1}^{r} \left[\prod_{i_\alpha=1}^{N_\alpha} \mathrm{d}^2 z_{i_\alpha}^{(\alpha)} \prod_{i_\alpha < j_\alpha}^{N_\alpha} |z_{i_\alpha}^{(\alpha)} - z_{j_\alpha}^{(\alpha)}|^{2(-1)^\alpha \theta_\alpha} \mathrm{e}^{-|q_\alpha| \sum_{i_\alpha} |z_{i_\alpha}^{(\alpha)}|^2} \right]$$
$$\times \mathrm{e}^{-\frac{1}{2}\sum_1^{N_0} |z_k|^2} \prod_{\beta=0}^{r} \prod_{i_\beta < j_\beta}^{N_\beta} (\tilde{z}_{i_\beta}^{(\beta)} - \tilde{z}_{j_\beta}^{(\beta)})^{p_\beta} \prod_{i_{\beta+1}, j_\beta = 1}^{N_{\beta+1}, N_\beta} (\bar{\tilde{z}}_{i_{\beta+1}}^{(\beta+1)} - \tilde{z}_{j_\beta}^{(\beta)}) \tag{27}$$

where $\tilde{z}_{i_\beta}^{(\beta)} = z_{i_\beta}^{(\beta)}$ for $\beta =$ even and $\tilde{z}_{i_\beta}^{(\beta)} = \bar{z}_{i_\beta}^{(\beta)}$ for $\beta =$ odd and

$$\theta_0 = 0 \qquad \theta_r = \frac{(-1)^r}{p_{r-1} - (-1)^r \theta_{r-1}}$$
$$q_0 = -1 \qquad q_r = (-1)^{r+1} q_{r-1} \theta_r.$$

Now, the filling factor is

$$\nu = \cfrac{1}{p_0 + \cfrac{1}{p_1 + \cfrac{1}{\cdots + \cfrac{1}{p_r}}}} \tag{28}$$

where p_0 is odd and the other p_i are even integers.

By generalizing the calculations of Laughlin given in [1] and using the scalar product defined in (10), one can show that ψ_ν states are orthogonal [11].

To emphasize the second quantized character of our construction let us introduce the states

$$|i, p\rangle_T = \int \mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_{N_0}\, \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^{N_0} |z_k|^2} \psi_{\frac{i}{2p+1}}(z_1, \ldots, z_{N_0}) |z_1, \ldots, z_{N_0}\rangle \tag{29}$$

where $i = 1, \ldots, 2p+1$; $p = 1, 2, \ldots$, so that any filling factor $\nu = P/Q$ is considered. We used the vectors (13) with N replaced by N_0. The subscript T denotes the fact that trial wave functions are used to give an explicit realization.

The states (29) are orthonormal:

$$_T\langle i, p | j, p' \rangle_T = \delta_{i,j} \delta_{p,p'}. \tag{30}$$

We have shown that the states $|i, p\rangle_T$ are orthonormal by using the explicit form of trial wavefunctions. However, this should be a universal feature of QHE wavefunctions. Then, even if we do not know the explicit form, we can say that exact states of the QHE which we indicate with $|i, p\rangle$, should be orthonormal:

$$\langle i, p | j, p' \rangle = \delta_{i,j} \delta_{p,p'}. \tag{31}$$

Indeed, in the following we will use this universal property of QHE states without referring to any trial wavefunction.

To generalize the construction given in section 3, let us deal with the states

$$|1, p\rangle, |2, p\rangle, \ldots, |2p, p\rangle, |2p+1, p\rangle \tag{32}$$

corresponding to the filling factors, respectively,

$$\nu = \frac{1}{2p+1}, \frac{2}{2p+1}, \ldots, \frac{2p}{2p+1}, 1. \tag{33}$$

Define the following second quantized operators acting in the space spanned by the states (32),

$$\tilde{K} = \sum_{i=1}^{2p+1} q^i |i, p\rangle\langle i, p| \tag{34}$$

$$\tilde{E}_+ = \sum_{i=1}^{2p+1} a_i |i, p\rangle\langle i+2, p| \tag{35}$$

$$\tilde{E}_- = \sum_{i=1}^{2p+1} \bar{a}_i |i+2, p\rangle\langle i, p| \tag{36}$$

where

$$q^{2p+1} = 1. \tag{37}$$

To obtain the compact forms we adopted the definitions

$$|2p+2, p\rangle \equiv |1, p\rangle, |2p+3, p\rangle \equiv |2, p\rangle.$$

By using the orthonormality condition (30) one observes that the inverse of $\tilde{K}$ is

$$\tilde{K}^{-1} = \sum_{i=1}^{2p+1} q^{-i} |i, p\rangle\langle i, p| = \tilde{K}^\dagger. \tag{38}$$

The coefficients a_i are nonzero and satisfy

$$|a_{2p+1}|^2 - |a_{2p-1}|^2 = 0$$
$$|a_{2p}|^2 - |a_{2p-2}|^2 = -1$$
$$|a_{l+2}|^2 - |a_l|^2 = \frac{q^{l+2} - q^{-l-2}}{q - q^{-1}}$$

where $l = -1, 0, \ldots, (2p-3)$; $a_{-1} \equiv a_{2p}$, $a_0 \equiv a_{2p+1}$. Then, in terms of the basis $(|1, p\rangle, \ldots, |2p+1, p\rangle)$ the operators (34)–(36) lead to a $(2p+1)$-dimensional unitary irreducible cyclic representation of $U_q(sl(2))$ at q satisfying (37).

Note that the Casimir operators are proportional to unity as before: $\tilde{K}^{2p+1} = \mathbf{1}$ and $\tilde{E}_+^{2p+1} = \tilde{E}_-^{\dagger 2p+1} = (\prod_{i=1}^{2p+1} a_i)\mathbf{1}$.

5. Discussions

It is shown that QHE wavefunctions can be classified as irreducible cyclic representations of $U_q(sl(2))$ at roots of unity in a very natural way. This naturalness follows from the fact that the most significant physical quantity of the QHE $\nu = P/Q$ fits very well with the integer (m in (2)) characterizing irreducible cyclic representations of $U_q(sl(2))$. Obviously, any set of orthogonal states possessing a quantum number which permits a partition of unity like ν,

$$\sum_{i=1}^{2p+1} \frac{\nu(|i, 2p+1\rangle)}{p+1} = 1$$

can be classified as irreducible cyclic representation of $U_q(sl(2))$ at a root of unity.

How can one utilize the proposed classification of the QHE to calculate some physical quantities? Here, one of the most significant physical quantities is the partition function which may be obtained if the Green function in the space defined by $U_q(sl(2))$ at roots of unity with cyclic representation is available. In [13] the Green function in the space defined by the q-deformed group $SU_q(2)/U(1)$ for q is not a root of unity, is obtained. We hope that a similar calculation can be used in our case. Then, we can obtain the Green function and in terms of that the related partition function which may give some hints about its physical interpretation which is not clear at the moment.

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